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ON THE THEORY OF GEOMETRIC OBJECTS

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Introduction

Since the beginning of this century attempts have been made to define a concept general enough to include all structures which appear meaningfully in local differential geometry. Tensors were not general enough, as connections, very popular since their discovery in 1918, were not tensors. The same applies to bundles of jets and many other more recent constructions.

The first definitions of these "geometric objects" were published in 1936–1937 by J. A. Schouten and J. Haantjes [14] and by A. Wundheiler [16]. Attempts to classify geometric objects have been made since; cf. J. Aczel and S. Gołąb [1] for further references. It was not until 1952 that, with A. Nijenhuis' thesis [8] a quite complete treatment of the theory appeared. In this work the approach is what we may call classical or numerical: an object at a point P of a manifold M is a correspondence which assigns to each coordinate system defined at P a set of N numbers called components. The emphasis is placed on the fact that if the object is a geometric object, the transformations of its components are representations of the groupoid of transformation elements.

New attempts to formulate the theory of geometric objects were made by Haantjes-Laman (1953) [4] and Kuiper-Yano (1955) [5], with their fiber bundle approach. In the first paper the space of the object and the group of transformations operating on it are of primary importance.

We present here a "functorial" approach based on Nijenhuis' "natural bundle" (1958) [9] and a refinement of it due to E. Calabi. This approach reflects the observation that most properties of geometric objects are more easily—and more naturally—deduced from the basic conceptual property of geometric objects than is possible from the extremely technical coordinate or bundle definitions in previous literature. This basic property is that fields of geometric objects (and the "bundles" in which these are sections) are carried along by local diffeomorphisms. Accordingly, we define a "natural structure" on a manifold M as a triple $(E, \pi, \boldsymbol{B})$ where $\pi: E \to M$ is the projection map from the "total" manifold E to M, subject to a number of conditions, the most important one of which is that $\boldsymbol{B}$ is a functor which "lifts" every local

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diffeomorphism in M to one of E. Initially (E, π) is not a fiber bundle over M, though local triviality follows easily. If $(E, \pi, \boldsymbol{B})$ is of finite order the structure group will be a group of jets of diffeomorphisms of the same order, yet our treatment does not *require* finite order. Instead, we prove in § 5 that finiteness of order *follows*, through only in a certain local sense.

The paper confines itself to the study of Lie derivatives of fields of geometric objects and thus parallels Chapter II of Nijenhuis' thesis. § 1 gives the basic definitions related to the lifting of local diffeomorphisms. Calabi's 'continuity' hypothesis ensures that families of diffeomorphisms in M lift to families in E. In § 2 we take families of diffeomorphisms generated by families of vector fields, and obtain a lifting process for families of vector fields on M to families of vector fields on E. This makes possible the lift of a vector field whose one-parameter group of diffeomorphisms is unknown but which is known to be an element of a family of vector fields. That is the case of a bracket of vector fields whose lift and properties are studied in § 3. In § 4 we define the Lie derivative of a field of geometric objects and prove several properties of it. § 5 deals with linear differential operators between vector bundles with the same base manifold M. We define two operators $\boldsymbol{B}\phi$ and $L\phi$ for a fixed field ϕ of geometric objects with domain $U \subset M$, and show that their orders on suitable open sets of M are finite and equal at every point. Some other properties of them are proved.

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1. Bundle of geometric objects

1.1. Notation. a. The manifolds we use in this work are all differentiable C^∞-manifolds; all maps are also C^∞. $C^\infty(M)$ is the ring of real valued functions on M. If μ is a fiber bundle, $C^\infty(\mu)$ is the set of sections of μ. If M is the base manifold of μ and U an open set of M, then $C^\infty(\mu \mid U)$ is the set of sections of μ with domain on U.

b. If M is a manifold, we denote by T_mM the tangent space to M at $m \in M$ and $TM = \bigcup_{m \in M} T_mM$. If $f\colon M \to N$ is a map, its differential is $df\colon TM \to TN$. For curves $\gamma\colon R \to M$ we write γ_* instead of $d\gamma$.

c. If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an n-tuple of nonnegative integers, we put $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. If $x = (x_1, \cdots, x_n)$ is an n-tuple of real numbers, then $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and we write D^α for the differential operator:

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} .$$

d. If μ is a differentiable fiber bundle, we denote by $J_p^k(\mu)$ the space of k-jets of μ at p, and by $j_p^k(f)$ the k-jet of f at $p, f \in C^\infty(\mu)$, ([12], [13]).

e. If M is a manifold we denote by $C(M)$ the category whose objects are the open sets of M and whose morphisms are the local diffeomorphisms between those objects.

1.2. Definition. A "natural structure" on a manifold M is given by a triple (E, π, B) where: E is a manifold, $E \neq \emptyset$; $\pi: E \to M$ is a C^∞-map (projection); $B: C(M) \to C(E)$ is a covariant functor called "natural functor" satisfying:

i) if $U \in \mathrm{Ob}\,(C(M))$, then $B(U) = \pi^{-1}(U) \in \mathrm{Ob}\,(C(E))$,

ii) if $f \in \mathrm{Mor}\,(C(M))$ with $f: U \to U'$, then $B(f): \pi^{-1}(U) \to \pi^{-1}(U')$, where Ob stands for object, Mor for morphism, and $B(f) \in \mathrm{Mor}\,(C(E))$ and satisfies:

a. if $f(x) = y$, then $B(f)(\pi^{-1}(x)) = \pi^{-1}(y)$, i.e., $\pi B(f) = f\pi$,

b. if $W \in \mathrm{Ob}\,(C(M))$, $W \subset U$, then $B(f)|\pi^{-1}(W) = B(f|W)$;

iii) if N is any manifold and $f_n: M \to M$ is a diffeomorphism for every $n \in N$ such that the map

$$\begin{aligned} H: N \times M &\longrightarrow N \times M \\ (n, m) &\dashrightarrow (n, f_n(m)) \end{aligned}$$

is also a diffeomorphism, then the map

$$\begin{aligned} H^*: N \times E &\longrightarrow N \times E \\ (n, e) &\dashrightarrow (n, B(f_n)(e)) \end{aligned}$$

is a diffeomorphism.

1.2.1. Remark. The projection π and the fibers over every point of M, in the definition above, have the following properties:

A. *π is surjective.* It is obvious from condition ii) and the fact that we can always get a local diffeomorphism which sends any point of M to any other point of M.

B. The fibers over every point of M are not empty (by property A) and isomorphic to each other.

C. *$d\pi$ is surjective.* It follows from the fact that if $v_m \in T_m M$, there always exists a vector field v defined in a neighbourhood U of m, with local one-parameter group of diffeomorphisms $\{f_t\}$ such that $v(m) = v_m$. Then by condition ii), $\{B(f_t)\}$ is a local one-parameter group of diffeomorphisms on $\pi^{-1}(U)$ such that if b is any element of $\pi^{-1}(m)$ and $h \in C_0^\infty(\pi^{-1}(U))$, then

$$V(b)(h) = \left.\frac{d}{dt}\right|_{t=0} hB((f_t)(b)$$

is a tangent vector of E at b. The projection $d\pi(V(b))$ is v_m by condition ii a).

D. As a consequence of the fact that $d\pi$ is surjective it is possible to get a C^∞-manifold structure on $\pi^{-1}(m)$ for every $m \in M$ such that if $i: \pi^{-1}(M) \to E$ is the identity map, $(\pi^{-1}(m), i)$ is a submanifold of E with $\dim \pi^{-1}(m) = \dim E - \dim M$, [15].

1.3. Definition. A manifold M together with a "natural structure" $(E, \pi, \boldsymbol{B})$ is a *bundle of geometric objects* denoted $(M; E, \pi, \boldsymbol{B})$. A section ϕ of $\pi: E \to M$ with domain on an open set U of M is a *field of geometric objects* on U and for every $x \in U$, $\phi(x)$ is a *geometric object* at x.

1.4. Given an element $f \in \mathrm{Mor}\,(\boldsymbol{C}(M))$, $f: U \to U'$, and a field ϕ of geometric objects with domain on U, we define a new field ϕ^* of geometric objects on U' by

$$\phi^*(x) = \boldsymbol{B}(f)\phi(f^{-1}(x)) , \qquad x \in U' .$$

That ϕ^* is a field of geometric objects on U' follows from the following commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\phi} & \pi^{-1}(U) \\ {\scriptstyle f^{-1}}\uparrow & & \downarrow{\scriptstyle \boldsymbol{B}(f)} \\ U' & \longrightarrow & \pi^{-1}(U') \end{array}$$

We say that ϕ^* is induced by ϕ by means of f.

1.5. Examples of bundles of geometric objects. Later we shall make use of some bundles of geometric objects derived from a given one, say $(M; E, \pi, \boldsymbol{B})$, and therefore it seems to be useful to study them now.

A. *Tangent bundle of M*. It is obviously a bundle of geometric objects where the functor

$$\boldsymbol{B}': \boldsymbol{C}(M) \to \boldsymbol{C}(T(M))$$

is given by

i) if $U \in \mathrm{Ob}\,(\boldsymbol{C}(M))$, then $\boldsymbol{B}'(U) = TU$,

ii) if $f \in \mathrm{Mor}\,(\boldsymbol{C}(M))$, then $\boldsymbol{B}'(f) = df$.

Clearly df satisfies ii a) and ii b) of Definition 1.2. We shall see that it also satisfies iii). Let

$$H: N \times M \to N \times M$$

with $H(n, m) = (n, f_n(m))$, be a diffeomorphism where N is any manifold and $f_n: M \to M$ is also a diffeomorphism for every $n \in N$. Then $df_n: TM \to TM$ is a diffeomorphism and so is $dH: T(N \times M) \to T(N \times M)$. Therefore the map

$$H^*: N \times TM \to N \times TM$$

which can be factored as follows:

$$\begin{array}{ccccccc} N \times TM & \longrightarrow & TN \times TM & \longrightarrow & TN \times TM & \longrightarrow & TN \times TM \\ (n, v(m)) & \dashrightarrow & (0(n), v(m)) & \dashrightarrow & (0(n), df_n v(m)) & \dashrightarrow & (n, df_n v(m)) , \end{array}$$

is a diffeomorphism.

B. If the projection $\pi^*: TE \to M$ be such that $\pi^*(v_e) = \pi(e)$, and the functor $\boldsymbol{B}^*: \boldsymbol{C}(M) \to \boldsymbol{C}(TE)$ be such that

i) $\boldsymbol{B}^*(U) = T(\pi^{-1}(U)), U \in \text{Ob}(\boldsymbol{C}(M))$,

ii) $\boldsymbol{B}^*(f) = d\boldsymbol{B}(f), f \in \text{Mor}(\boldsymbol{C}(M))$,

then $(M; TE, \pi^*, \boldsymbol{B}^*)$ is a bundle of geometric objects.

C. Let $(M; E, \pi, \boldsymbol{B})$ be a bundle of geometric objects. By Remark 1.2.1 we know that for every $x \in M, \pi^{-1}(x)$ has a unique manifold structure such that $(\pi^{-1}(x), i)$ is a submanifold of E (i: the inclusion map). Moreover if $\dim E = m$, $\dim M = n$, then $\dim \pi^{-1}(x) = m - n$. Obviously $T(\pi^{-1}(x))$ is also a $2(m-n)$-dimensional manifold. Therefore

$$X = \bigcup_{x \in M} T(\pi^{-1}(x))$$

is a $(2m - n)$-dimensional closed submanifold of TE. We may then build a new bundle of geometric objects on M with the data $(X, \pi', \boldsymbol{B}')$, where:

$$\pi': X \to M$$

satisfies $\pi'(v_e) = \pi(e)$. This means that π' is the composition of the restriction of $p': TE \to E$ to X and π, both of which are C^∞-maps.

$$\boldsymbol{B}': \boldsymbol{C}(M) \to \boldsymbol{C}(X)$$

is a covariant functor which satisfies:

i) if $U \in \text{Ob}(\boldsymbol{C}(M))$, then $\boldsymbol{B}'(U) = \pi'^{-1}(U) = \bigcup_{x \in U} T(\pi^{-1}(x))$,

ii) if $f \in \text{Mor}(\boldsymbol{C}(M))$, then $\boldsymbol{B}'(f) = d\boldsymbol{B}(f)$.

Conditions ii a) and ii b) of Definition 1.2 are obviously satisfied. We have to prove that condition iii) is also satisfied. Let N be any manifold and let

$$H: N \times M \to N \times M$$

be a diffeomorphism such that $H(n, x) = (n, f_n(x))$ with $f_n: M \to M$ a diffeomorphism for every $n \in N$. Therefore, as $(M; E, \pi, \boldsymbol{B})$ is a bundle of geometric objects,

$$H^*: N \times E \to N \times E$$

with $H^*(n, e) = (n, \boldsymbol{B}(f_n)(e))$ is a diffeomorphism and so is dH^*. Then the map

$$H^{**}: N \times X \to N \times X$$

with $H^{**}(n, v_e) = (n, d\boldsymbol{B}(f_n)(v_e))$, which can be factored as follows:

$$N \times X \longrightarrow N \times TE \longrightarrow TN \times TE \xrightarrow{dH^*} TN \times TE \longrightarrow N \times TE \longrightarrow N \times X$$

$$(n, v_e) \dashrightarrow (n, v_e) \dashrightarrow (0_n, v_e) \dashrightarrow (0_n, dB(f_n)v_e) \dashrightarrow (n, dB(f_n)v_e) \dashrightarrow (n, dB(f_n)v_e)$$

is a diffeomorphism. Therefore $(M; X, \pi', B')$ is a subbundle of geometric objects of $(M; TE, \pi^*, B^*)$.

2. The lift of a family of vector fields in a bundle of geometric objects

2.1. Let W be an open set of $R \times M$, where R is the real line and M is a manifold. A family $\{v_t\}$ of vector fields on $\pi_2(W) \subset M$ is a map [10]

$$v: W \to T(\pi_2(W))$$

with $v(t, x) = v_t(x)$, C^1 at t and C^∞ at x, which makes commutative the following diagram:

$$\begin{array}{ccc} W & \xrightarrow{v} & T(\pi_2(W)) \\ & {\scriptstyle \pi_2}\searrow \quad \swarrow{\scriptstyle p} & \\ & M & \end{array}$$

where p is the projection of $TM \to M$.

The theory of differential equations says that every pair $(s, y) \in W$ determines an integral curve $c_{s,y}$ of $\{v_t\}$ defined for every t of an open neighbourhood of $s \in R$, such that

$$(0) \qquad v_t(c_{s,y}(t)) = c_{s,y*}(d/dr) \quad \text{and} \quad c_{s,y}(s) = y\,,$$

where d/dr is the unit vector at the origin of R. These integral curves have, among others, the following properties:

a. each integral curve belongs to a maximal integral curve;

b. two integral curves have a point of $W \subset R \times M$ in common if and only if they are joinable (if they have the same value at a common point of their domains they have the same value at every point of the intersection of their domains, and the union of the curves is the curve defined on the union of those domains).

The family $\{v_t\}$ of vector fields defines also a map

$$f: R \times R \times M \to M$$

given by

$$f(s, t, y) = c_{s,y}(t) ,$$

where $s \in \pi_1(W) \subset R$ and $t \in \operatorname{dom} c_{s,y}$ (maximal). For each fixed pair (s, t), f defines a local diffeomorphism on M

$$f(s, t, y) = f_{s,t}(y)$$

and, for each fixed s, a family $\{f_t\}$ of local diffeomorphisms on M, which we say is generated by the family $\{v_t\}$. The map f has the following properties:

c) $f_{t,r}(f_{s,t}(y)) = f_{s,r}(y)$,

d) $f_{s,t} = f_{t,s}^{-1}$,

e) $f_{s,s} = \mathrm{id}$ for every s.

If the family $\{v_t\}$ of vector fields does not depend on the parameter t it becomes a unique vector field $v = v_t$ for all t, defined on an open set U of M, and therefore the family $\{f_t\}$ of local diffeomorphisms generated by $\{v_t\}$ satisfies the condition

$$f_{s,t} = f_{0,t-s} .$$

In fact, as the curve $c_{0,y}$ is a reparametrization of $c_{s,y}$, both curves are solutions by y of the same differential equation because $v_t = v_{t-s}$. Then

$$c_{s,y}(t) = c_{0,y}(t - s) .$$

If we put $f_t = f_{0,t}$, the family $\{f_t\}$ satisfies:

c') $f_s f_t = f_{s+t}$ for all s, t such that $s + t \in \operatorname{dom}(c_{0,x})$, $x \in U$,

d') $f_{-s} = f_s^{-1}$,

e') $f_0 = \mathrm{id}$.

Therefore, when the family $\{v_t\}$ consists of a unique element v which does not depend on t, it generates a one-parameter group $\{f_t\}$ of local diffeomorphisms.

Then, given a family $\{v_t\}$ of vector fields with domain on an open set U of a manifold M, and given a function $k \in C^\infty(U)$, we can write, by (0),

$$v_{t_0}(f_{s,t_0}(x))(k) = \frac{d}{dt}\bigg|_{t=t_0} k f_{s,t}(x) ,$$

which we shall write omitting s when it does not lead to confusion. Moreover, considering that $f_{t_0,t_0} = \mathrm{id}$ and $f_{t_0,t} = f_{s,t} f_{s,t_0}^{-1}$, we obtain the more convenient formula:

$$v_{t_0}(x)(k) = \frac{d}{dt}\bigg|_{t=t_0} k f_{t_0,t}(x) = \frac{d}{dt}\bigg|_{t=t_0} k f_{s,t} f_{s,t_0}^{-1} ,$$

or omitting s

$$(1) \qquad v_{t_0}(x)(k) = \frac{d}{dt}\bigg|_{t=t_0} kf_t f_{t_0}^{-1}(x) \;.$$

If the family $\{v_t\}$ does not depend on the parameter t, the above formula becomes:

$$v(f_t(x))(k) = \frac{d}{dt}\bigg|_{t} kf_t(x) \;,$$

or

$$(2) \qquad v(x)(k) = \frac{d}{dt}\bigg|_{t=0} kf_t(x) \;.$$

2.2. Let $(M; E, \pi, B)$ be a bundle of geometric objects, $\{v_t\}$ a family of vector fields defined on an open subset U of M, and $\{f_t\}$ the family of local diffeomorphisms generated by $\{v_t\}$. It is possible, by means of the functor B, to define a new family $\{B(f_t)\}$ of local diffeomorphisms with domain on the open subset $\pi^{-1}(U)$ of E. This family defines in turn a family of vector fields on $\pi^{-1}(U)$, given by

$$(3) \qquad B(v)_{t_0}(b)(h) = \frac{d}{dt}\bigg|_{t=t_0} hB(f_{t_0,t})(b) = \frac{d}{dt}\bigg|_{t=t_0} hB(f_t)B(f_{t_0}^{-1})(b) \;,$$

where, in the third member, the parameter s has been omitted, $h \in C^\infty(\pi^{-1}(U))$, and $b \in \pi^{-1}(U)$.

If the family $\{v_t\}$ does not depend on the parameter t, formula (3) becomes (as we saw in Remark 1.2.1)

$$(4) \qquad B(v)(b)(h) = \frac{d}{dt}\bigg|_{t=0} hB(f_t)(b) \;.$$

2.3. Lemma. *The family $\{B(v)_t\}$ of local diffeomorphisms on $\pi^{-1}(U)$ defined above has the following properties:*

a) *If $\{v_t\}$ and $\{v'_t\}$ are two families of vector fields on U, then*

$$B(v + v')_t = B(v)_t + B(v')_t \;.$$

b) *If $\alpha: R \to R$ is a C^∞-function, and $\{v_t\}$ a family of vector fields on U, then*

$$B(\alpha \cdot v)_t = \alpha(t) \cdot B(v)_t \;.$$

Proof. a) If $\{f_t\}$ and $\{f'_t\}$ are the families of local diffeomorphisms generated by $\{v_t\}$ and $\{v'_t\}$ respectively, and $k \in C^\infty(U)$, we have:

$$\left.\frac{d}{dt}\right|_{t=t_0} kf_{t_0,t}f'_{t_0,t}(x) = \left.\frac{d}{dr}\right|_{r=t_0} kf_{t_0,r}(x) + \left.\frac{d}{ds}\right|_{s=t_0} kf'_{t_0,s}(x)$$
$$= (v_{t_0}(x) + v'_{t_0}(x))(k) .$$

Then by putting $f_{t_0,t}f'_{t_0,t} = f^*_{t_0,t}$,

$$(v_{t_0} + v'_{t_0})(x)(k) = \left.\frac{d}{dt}\right|_{t=t_0} kf^*_{t_0,t}(x)$$

and applying formula (3) of § 2.2, we have for $b \in \pi^{-1}(U)$ and $h \in C^\infty(\pi^{-1}(U))$,

$$\begin{aligned} \boldsymbol{B}(v + v')_{t_0}(b)(h) &= \left.\frac{d}{dt}\right|_{t=t_0} h\boldsymbol{B}(f^*_{t_0,t})(b) \\ &= \left.\frac{d}{dt}\right|_{t=t_0} h\boldsymbol{B}(f_{t_0,t})\boldsymbol{B}(f'_{t_0,t})(b) \\ &= \left.\frac{d}{dr}\right|_{r=t_0} h\boldsymbol{B}(f_{t_0,t})(b) + \left.\frac{d}{ds}\right|_{s=t_0} h\boldsymbol{B}(f'_{t_0,s})(b) \\ &= (\boldsymbol{B}(v)_{t_0} + \boldsymbol{B}(v')_{t_0})(b)(h) . \end{aligned}$$

b) Let $\alpha: R \to R$ be a C^∞-function. It can be considered as a vector field on R, which defines a one-parameter group of local diffeomorphisms $\{g_t\}$ such that

$$\alpha(g_t(t_0)) = \left.\frac{d}{dt}\right|_{t} g_t(t_0) ,$$

and in particular

$$\alpha(t_0) = \left.\frac{d}{dt}\right|_{t=0} g_t(t_0) .$$

Let $k \in C^\infty(U)$, and let $\{f_t\}$ be the family of local diffeomorphisms generated by $\{v_t\}$. Then

$$\left.\frac{d}{dt}\right|_{t=0} kf_{t_0,g_t(t_0)}(x) = \left.\frac{d}{dg_t(t_0)}\right|_{g_t(t_0)=t_0} kf_{t_0,g_t(t_0)}(x) \cdot \left.\frac{d}{dt}\right|_{t=0} g_t(t_0)$$
$$= \alpha(t_0) \cdot v_{t_0}(x)(k) = (\alpha \cdot v)_{t_0}(x)(k) .$$

This result and formula (3) of § 2.2 lead us to the following expression, where $b \in \pi^{-1}(U)$ and $h \in C^\infty(\pi^{-1}(U))$:

$$\boldsymbol{B}(\alpha \cdot v)_{t_0}(b)(h) = \left.\frac{d}{dt}\right|_{t=0} h\boldsymbol{B}(f_{t_0,g_t(t_0)})(b)$$

$$= \frac{d}{dg_t(t_0)}\bigg|_{g_t(t_0)=t_0} hB(f_{t_0,g_t(t_0)})(b) \cdot \frac{d}{dt}\bigg|_{t=0} g_t(t_0)$$
$$= \alpha(t_0)B(v)_{t_0}(b) \ .$$

2.4. We claim that the family $\{B(v)_t\}$ is well defined, that is, if $\{v_t\}$ and $\{v'_t\}$ are two families of vector fields with domain U such that $v_0 = v'_0$ on an open subset $V \subset U$, then $B(v)_0 = B(v')_0$ on $\pi^{-1}(V)$. Or, considering the properties of $\{B(v)_t\}$ proved above, if the element v_0 of $\{v_t\}$ is null on $V \subset U$, then $B(v)_0 = 0$ on $\pi^{-1}(V)$. In order to prove it, we need the following lemma.

2.4.1. Lemma. *Let M be a manifold, and $\{v_t\}$ be a family of vector fields defined on an open subset U of $R \times M$ such that*

$$v(t, x) = v_t(x) \qquad \textit{for each fixed } t \ ,$$

and

$$v(0, x) = v_0(x) = 0 \qquad \textit{for all } x \in \pi_2(U) \ .$$

Then there exists another family $\{w_t\}$ of vector fields with the same domain of $\{v_t\}$ such that

$$v_t(x) = tw_t(x) \qquad \textit{for each } t \ .$$

(for the proof see [11, p. 8]).

Therefore, returning to our claim, we see that there is a family of vector fields $\{w_t\}$ with domain on V such that $\{v_t\} = \{tw_t\}$, and in particular $v_0 = 0 \cdot w_0 = 0$. Then, putting

$$\{v_t\} = \{tw_t\} = \{(\mathrm{id}\ w)_t\} \ ,$$

by property b) of Lemma 2.3 we have

$$\{B(v)_t\} = \{B(\mathrm{id}\ w)_t\} = \{tB(w)_t\} \ ,$$
$$B(v)_0 = 0 \cdot B(w)_0 = 0 \qquad \text{on } \pi^{-1}(V) \ .$$

3. The lift of a bracket of vector fields in a bundle of geometric objects

3.1. Let $(M; E, \pi, B)$ be a bundle of geometric objects, and let u and v be two vector fields defined on an open subset U of M. If $\{g_t\}$ and $\{f_t\}$ are the one-parameter groups of local diffeomorphisms generated by u and v respectively, it is possible to define a family of local diffeomorphisms on U, by putting for each t:

$$h_t = g_{-\sqrt{t}} f_{-\sqrt{t}} g_{\sqrt{t}} f_{\sqrt{t}} \ .$$

The family $\{h_t\}$ generates a family $\{w_t\}$ of vector fields on U, where $w_0 = [v, u]$, [2, p. 18]. That is, if $k \in C^\infty(U)$, considering formula (1) of § 2.1 and the fact that $h_0 = \mathrm{id}$ we have

$$w_0(x)(k) = \frac{d}{dt}\bigg|_{t=0} kh_t(x) = [v, u](x)(k) .$$

Then formula (3) of § 2.1 gives, for $k' \in C^\infty(\pi^{-1}(U))$ and $b \in \pi^{-1}(U)$,

$$\begin{aligned} \boldsymbol{B}([v, u])(b)(k') &= \boldsymbol{B}(w)_0(b)(k') \\ &= \frac{d}{dt}\bigg|_{t=0} k'\boldsymbol{B}(h)_t(b) \\ &= \frac{d}{dt}\bigg|_{t=0} k'\boldsymbol{B}(g_{-\sqrt{t}})\boldsymbol{B}(f_{-\sqrt{t}})\boldsymbol{B}(g_{\sqrt{t}})\boldsymbol{B}(f_{\sqrt{t}})(b) \\ &= [\boldsymbol{B}(v), \boldsymbol{B}(u)](b)(k') . \end{aligned}$$

Therefore

$$(5) \qquad [\boldsymbol{B}(v), \boldsymbol{B}(u)] = \boldsymbol{B}([v, u]) .$$

4. The Lie derivative of a field of geometric objects

4.1. Let v be a vector field defined on an open subset U of a manifold M. We may consider v as the element v_0 of a family $\{v_t\}$ of vector fields such that if $\{f_t\}$ is the family of local diffeomorphisms generated by $\{v_t\}$, then $f_0 = \mathrm{id}$. In fact, if v is the element v_{t_0} of a family $\{v_t\}$ which generates a family $\{f_t\}$ of local diffeomorphisms, we define a new family $\{g_t\}$ of local diffeomorphisms by putting for each t

$$g_t = f_{t+t_0} f_{t_0}^{-1} ,$$

which implies $g_0 = \mathrm{id}$. Let $\{v'_t\}$ be the family of vector fields which generates $\{g_t\}$. Then for each $x \in U$ we have $v'_0(x) = v_{t_0}(x) = v(x)$.

4.2. Let v be a vector field defined on an open subset U of a manifold M, the base of a bundle of geometric objects $(M; E, \pi, \boldsymbol{B})$, and suppose that v is the element v_0 of a family $\{v_t\}$ of vector fields on U which generates a family $\{f_t\}$ of local diffeomorphisms on U with $f_0 = \mathrm{id}$. If ϕ is a field of geometric objects on U, and for $x \in U$, $U_x \subset U$ is a neighbourhood of x such that for some $\varepsilon > 0$

$$f_t(U_x) \subset U \qquad \text{for } |t| < \varepsilon ,$$

then we may define the curve

$$\gamma_x \colon (-\varepsilon, \varepsilon) \to \pi^{-1}(U)$$

given by

$$\gamma_x(t) = \boldsymbol{B}(f_t^{-1})\phi_{f_t(x)} \in \pi^{-1}(x) .$$

The curve γ_x is a C^∞-curve because it is the composition of the following C^∞-maps:

$$\begin{aligned} (-\varepsilon, \varepsilon) &\longrightarrow (-\varepsilon, \varepsilon) \times U \longrightarrow (-\varepsilon, \varepsilon) \times \pi^{-1}(U) \\ &\qquad \xrightarrow{c} (-\varepsilon, \varepsilon) \times \pi^{-1}(U) \longrightarrow \pi^{-1}(U) , \\ t &\dashrightarrow (t, f_t(x)) \dashrightarrow (t, \phi_{f_t(x)}) \\ &\qquad \dashrightarrow (t, \boldsymbol{B}(f_t^{-1})\phi_{f_t(x)}) \dashrightarrow \boldsymbol{B}(f_t^{-1})\phi_{f_t(x)} , \end{aligned}$$

where the map c is C^∞ due to condition iii) in the definition of $\boldsymbol{B}$.

4.3. Definition. Let ϕ be a field of geometric objects, and let v be a vector field, both defined on an open subset U of the base manifold M of a bundle of geometric objects such that they satisfy the conditions established in § 4.2. The *Lie derivative* at $x \in U$ of a field ϕ of geometric objects with respect to a vector field v is given by

$$(6) \qquad \underset{v}{L}\, \phi_x(h) = \frac{d}{dt}\bigg|_{t=0} h\boldsymbol{B}(f_t^{-1})\phi_{f_t(x)} ,$$

where $h \in C^\infty(\pi^{-1}(U))$. If v is an element v_{t_0} of a family $\{v_t\}$ of vector fields, by § 4.1, (1) becomes

$$(7) \qquad \underset{v}{L}\, \phi_x(h) = \frac{d}{dt}\bigg|_{t=t_0} h\boldsymbol{B}(f_{t_0}f_t^{-1})\phi_{f_t f_{t_0}^{-1}(x)} .$$

4.4. The Lie derivative $\underset{v}{L}\, \phi_x$ of a field ϕ of geometric objects with respect to a vector field v is well defined. That is, if there are two families of vector fields $\{v_t\}$ and $\{v'_t\}$ such that $v = v_0 = v'_0$, then

$$(8) \qquad \underset{v_0}{L}\, \phi_x = \underset{v'_0}{L}\, \phi_x .$$

In order to prove this, we need the following lemma in the theory of families of vector fields, [2], [9]:

4.4.1. Lemma. *Let $\{v_t\}$ and $\{u_t\}$ be two families of vector fields defined on an open subset U of a manifold M such that they generate the families $\{f_t\}$ and $\{f_t^{-1}\}$ of local diffeomorphisms, respectively. Then the following equality holds*

$$(9) \qquad v_t = -df_t u_t .$$

Proof. It follows from § 2.1 that the family $\{v_t\}$ of vector fields defines a map $f\colon R \times R \times M \to M$ by $f(s, t, y) = c_{s,y}(t)$. If we fix s and omit it, as we have been doing up to now, we may consider f as a map

$$f\colon R \times M \to M\ ,$$

which for every fixed t is given by

$$f(t, y) = f_t(y)\ .$$

Therefore, if we put $g_t = f_t^{-1}$ for $|t| < \varepsilon$, ε determined by $\{v_t\}$, then the maps

$$f\colon (-\varepsilon, \varepsilon) \times U \to U\ ,$$
$$g\colon (-\varepsilon, \varepsilon) \times U \to U\ ,$$

$$df\colon T((-\varepsilon, \varepsilon) \times U) \to TU\ ,$$
$$dg\colon T((-\varepsilon, \varepsilon) \times U) \to TU$$

satisfy

$$g(t, f(t, m)) = m\ ,$$
$$dg((r, t), df((r, t), (m, v))) = v(m)\ ,$$

and, in particular,

$$df((r, t), (m, 0)) = t\cdot v_r(f_r(m))\ .$$

Therefore

$$\begin{aligned} 0_m &= dg((r, t), df((r, t), (m, 0))) = dg((r, t), (f_r(m), tv_r)) \\ &= dg((r, 0), (f_r(m), tv_r)) + dg((r, t), (f_r(m), 0)) \\ &= dg_r(tv_r(f_r(m))) + tu_r(g_r(f_r(m)))\ , \end{aligned}$$

which implies

$$dg_r(v_r(f_r(m))) = -u_r(m)\ ,$$

or

$$(df_r^{-1}v_r)(m) = -u_r(m)\ .$$

Returning to our claim we see that if $\{f_t\}$ and $\{f'_t\}$ are families of the local diffeomorphisms generated by $\{v_t\}$ and $\{v'_t\}$ respectively, what we have to prove is that for $h \in C^\infty(\pi^{-1}(U))$,

$$\frac{d}{dt}\bigg|_{t=0} h\boldsymbol{B}(f_t^{-1})\phi_{f_t(x)} = \frac{d}{dt}\bigg|_{t=0} h\boldsymbol{B}(f_t'^{-1})\phi_{f'_t(x)}\ .$$

But the left hand side of this equality can be written as

$$\frac{d}{dt}\bigg|_{t=0} hB(f_t^{-1})\phi_{f_t(x)} = \frac{d}{dt}\bigg|_{t=0} hB(f_t^{-1})\phi_x + \frac{d}{dt}\bigg|_{t=0} h\phi_{f_t(x)} ,$$

and the right hand side can be written in a similar way. If, as in the above lemma, $\{u_t\}$ and $\{u'_t\}$ are the families of the vector fields which generate $\{f_t^{-1}\}$ and $\{f_t'^{-1}\}$ respectively, then we have

$$\text{(10)}\qquad \begin{aligned}\frac{d}{dt}\bigg|_{t=0} hB(f_t^{-1})\phi_{f_t(x)} &= B(u_0)\phi_x + d\phi v_0(x) \qquad \text{(by the lemma)}\\ &= d\phi v_0(x) - B(v_0)\phi_x ,\end{aligned}$$

$$\text{(11)}\qquad \begin{aligned}\frac{d}{dt}\bigg|_{t=0} hB(f_t'^{-1})\phi_{f'_t(x)} &= B(u'_0)\phi_x + d\phi v_0(x) \qquad \text{(by the lemma)}\\ &= d\phi v'_0(x) - B(v'_0)\phi_x .\end{aligned}$$

As $v_0 = v'_0$, it follows, by § 2.4., that (10) and (11) imply (8).

4.5. We obtain two important conclusions from §§ 4.3 and 4.4.:

a) Formulas (6) and (7) show that $\underset{v}{L}\,\phi_x$ is a tangent vector to the curve $\gamma_x \subset \pi^{-1}(x)$ at the point ϕ_x. This means that $\underset{v}{L}\,\phi_x \in T_{\phi_x}E$. Moreover, by observing that $\underset{v}{L}\,\phi_x(h) = 0$ for each function $h \in C^\infty(E)$, which is constant on $\pi^{-1}(x)$, we conclude that

$$\underset{v}{L}\,\phi_x \in T(\pi^{-1}(x)) .$$

Therefore $\underset{v}{L}\,\phi_x$ is, at each point where it is defined, a geometric object of the bundle $(M; X, \pi', B')$ (see § 1.5 C) and a subobject of $(M; TE, \pi^*, B^*)$ (see § 1.5 B).

b) Formula (10) is an important result in the theory of the Lie derivative of a geometric object, which can be generalized by means of Lemma 4.4.1 and formula (7). As we shall make frequent use of it, it will be useful to establish the following proposition.

4.5.1. Proposition. *Let v be a vector field defined on an open subset U of the base manifold of a bundle of geometric objects, and ϕ be a field of geometric objects defined on U. Then for every $x \in U$ we have*

$$\text{(12)}\qquad \underset{v}{L}\,\phi_x = d\phi(v)(x) - B(v)\phi_x .$$

If v is an element v_{t_0} of a family $\{v_t\}$ of the vector fields which generates the family $\{f_t\}$ of local diffeomorphisms on U, we have

$$\text{(13)}\qquad \underset{v_{t_0}}{L}\,\phi_x = d\phi(df_{t_0}^{-1}v_{t_0})(x) - dB(f_{t_0}^{-1})B(v)_{t_0}\phi_x .$$

4.6. Proposition. *Let $(M; E, \pi, \boldsymbol{B})$ be a bundle of geometric objects, ϕ be a field of geometric objects defined on an open subset U of M, $f\colon U \to f(U) \subset M$ a diffeomorphism, and v be a vector field on U. Then the following expression is satisfied:*

$$d\boldsymbol{B}(f) \underset{v}{L} \phi_x = \underset{dfv}{L} \boldsymbol{B}(f)\phi_x \,.$$

Proof. Assume that v is the element v_0 of a family of vector fields such that the family $\{f_t\}$ of local diffeomorphisms generated by v satisfies the condition $f_0 = \mathrm{id}$. Then, for $h \in C^\infty(\pi^{-1}(U))$,

$$\begin{aligned}(\underset{dfv}{L} \boldsymbol{B}(f)\phi_x)(h) &= \frac{d}{dt}\bigg|_{t=0} h\boldsymbol{B}(ff_t^{-1}f^{-1})\boldsymbol{B}(f)\phi_{ff_tf^{-1}(x)} \\ &= \frac{d}{dt}\bigg|_{t=0} (h\boldsymbol{B}(f))\boldsymbol{B}(f_t^{-1})(\phi f)_{f_t(f^{-1}(x))} \\ &= \underset{v}{L} (\phi f)_{f^{-1}(x)}(h\boldsymbol{B}(f)) = (d\boldsymbol{B}(f) \underset{v}{L} \phi_x)(h) \,.\end{aligned}$$

4.7. Corollary. *If the diffeomorphism f of the above proposition is an element f_t of the one-parameter group of diffeomorphisms generated by the vector field v, then, for any $h \in C^\infty(\pi^{-1}(U))$,*

$$d\boldsymbol{B}(f_{-t}) \underset{v}{L} \phi_{f_t(x)}(h) = \underset{v}{L} (\boldsymbol{B}(f_{-t}))\phi_{f_t(x)}(h) = \frac{d}{dt}\bigg|_t h\boldsymbol{B}(f_{-t})\phi_{f_t(x)} \,.$$

Proof.

$$\begin{aligned}\frac{d}{dt}\bigg|_t h\boldsymbol{B}(f_{-t})\phi_{f_t(x)} &= \frac{d}{ds}\bigg|_{s=0} h\boldsymbol{B}(f_{-s-t})\phi_{f_{s+t}(x)} \\ &= \frac{d}{ds}\bigg|_{s=0} h\boldsymbol{B}(f_{-s})\boldsymbol{B}(f_{-t})\phi_{f_sf_t(x)} \\ &= \underset{v}{L} \boldsymbol{B}(f_{-t})\phi_{f_t(x)}(h) = d\boldsymbol{B}(f_{-t}) \underset{df_tv}{L} \phi_{f_t(x)} \\ &= d\boldsymbol{B}(f_{-t}) \underset{v}{L} \phi_{f_t(x)} \,,\end{aligned}$$

because $(df_tv)(x) = v(x)$ for each t.

4.8. Proposition. *Let ϕ be a field of geometric objects defined on an open subset U of a manifold M, and let v, u be two vector fields with domain on U. Then for any pair (a, b) of real numbers the following expression is satisfied:*

$$\underset{av+bu}{L} \phi_x = a \underset{v}{L} \phi_x + b \underset{u}{L} \phi_x \,.$$

Proof. It is obvious from Proposition 4.5.1, the linearity of $d\phi$, and the operator $v \to \boldsymbol{B}(v)$.

4.9. Definition. Let $(M; E, \pi, B)$ be a bundle of geometric objects, and $\{f_t\}$ a family of local diffeomorphisms defined on an open set $(-\varepsilon, \varepsilon) \times U \subset R \times M$. A field ϕ of geometric objects is invariant under the deformation $\{f_t\}$ if:

a. $\operatorname{dom}(f_t) \subset \operatorname{dom}(\phi)$, $\operatorname{im}(f_t) \subset \operatorname{im}(\phi)$ for each $|t| < \varepsilon$,

b. $B(f_t)\phi_x = \phi_{f_t(x)}$ for each $x \in U$ and $|t| < \varepsilon$.

4.9.1. Corollary. *If a field of geometric objects is invariant under $\{f_t\}$, it is also under $\{f_t^{-1}\}$.*

4.10. Proposition. *Let $(M; E, \pi, B)$ be a bundle of geometric objects, and $\{v_t\}$ be a family of vector fields on an open subset U of M generating a family $\{f_t\}$ of diffeomorphisms. Then a field ϕ of geometric objects defined on U is invariant under the deformation $\{f_t\}$ if and only if $\underset{v}{L}\,\phi_x = 0$ for each $x \in U$ and $|t| < \varepsilon$.*

Proof. If ϕ is invariant under the deformation $\{f_t\}$, it is invariant under $\{f_t^{-1}\}$. Thus, if $h \in C^\infty(\pi^{-1}(U))$, we have

$$\underset{v_{t_0}}{L}\,\phi_x(h) = \frac{d}{dt}\bigg|_{t=t_0} hB(f_{t_0,t}^{-1})\phi_{f_{t_0,t}(x)} = \frac{d}{dt}\bigg|_{t=t_0} h\phi_x = 0\ .$$

Conversely, if $\underset{v}{L}\,\phi_x = 0$ for each $x \in U$ and $|t| < \varepsilon$, then $\underset{v_{t_0}}{L}\,\phi_{f_{t_0}(x)} = 0$ for $|t_0| < \varepsilon$ and, by linearity, $dB(f_{t_0}^{-1})\underset{v_{t_0}}{L}\,\phi_{f_{t_0}(x)} = 0$. Therefore for any $h \in C^\infty(\pi^{-1}(U))$ we have

$$\begin{aligned}
0 &= \underset{v_{t_0}}{L}\,\phi_{f_{t_0}(x)}(hB(f_{t_0}^{-1})) \\
&= \frac{d}{dt}\bigg|_{t=t_0} hB(f_{t_0}^{-1})B(f_{t_0}f_t^{-1})\phi_{f_t f_{t_0}^{-1}(f_{t_0}(x))} \\
&= \frac{d}{dt}\bigg|_{t=t_0} hB(f_t^{-1})\phi_{f_t(x)}\ ,
\end{aligned}$$

which implies that the curve $\gamma_x(t) = B(f_t^{-1})\phi_{f_t(x)}$ has vanishing tangent at each point. Thus the curve is constant so that

$$B(f_t^{-1})\phi_{f_t(x)} = \phi_x\ , \quad \text{or} \quad \phi_{f_t(x)} = B(f_t)\phi_x\ .$$

4.11. Lemma. *Let $(M; E, \pi, B)$ be a bundle of geometric objects, and ϕ a field of geometric objects on an open subset U of M. If the families of the vector fields $\{v_t\}$ and $\{u_t\}$ satisfy the condition*

$$\underset{v_t}{L}\,\phi_x = \underset{u_t}{L}\,\phi_x = 0$$

for each $x \in U$ and $|t| < \varepsilon$, and $\{f_t\}$ is the family of local diffeomorphisms generated by $\{v_t\}$, then we have

$$\underset{df_s u_t}{L} \phi_x = 0 \qquad \textit{for } |t| < \varepsilon, |s| < \varepsilon ,$$

and, in particular, $\underset{df_s v_t}{L} \phi_x = 0$ *for* $|t| < \varepsilon$, $|s| < \varepsilon$.

Proof. If $\underset{v_t}{L} \phi_x = 0$ for each $x \in U$ and $|t| < \varepsilon$, and ϕ is invariant under the deformations $\{f_t\}$ and $\{f_t^{-1}\}$, then

$$\begin{aligned}\underset{df_s u_t}{L} \phi_x &= \underset{df_s u_t}{L} B(f_s f_s^{-1}) \phi_{f_s f_s^{-1}(x)} \\ &= dB(f_s) \underset{u_t}{L} B(f_s^{-1}) \phi_{f_s f_s^{-1}(x)} \\ &= dB(f_s) \underset{u_t}{L} \phi_{f_s^{-1}(x)} = 0 .\end{aligned}$$

4.12. Lemma. *Let* $(M; E, \pi, B)$ *be a bundle of geometric objects, and* ξ *the tangent bundle of* M: $\xi = (TM; p, M)$. *If* ϕ *is a field of geometric objects defined on an open subset* U *of* M, *then the set*

$$G(U, \phi) = \{v;\ v \in C^\infty(\xi \mid U),\ \underset{v}{L} \phi_x = 0,\ \textit{for each } x \in U\}$$

is a vector subspace of $C^\infty(\xi \mid U)$, *and the set*

$$\mathscr{G}(U, \phi, \varepsilon) = \{\{f_t\};\ \{f_t\} \textit{ generated by } \{v_t\};\ v_t \in G(U, \phi),\ |t| < \varepsilon\}$$

is a group under the composition $\{f_t\} \cdot \{g_t\} = \{f_t \cdot g_t\}$, $|t| < \varepsilon$.

Proof. 1. a) $\bar{O} \in G(U, \phi)$.

b) If $v \in G(U, \phi)$, then $-v \in G(U, \phi)$.

c) If $v, u \in G(U, \phi)$ and $a, b \in R$, then $av + bu \in G(U, \phi)$.

2. a) The family $\{f_t\}$ with $f_t = \text{id}$ for each t belongs to $\mathscr{G}(U, \phi, \varepsilon)$ because $\bar{O} \in G(U, \phi)$.

b) If $\{f_t\} \in \mathscr{G}(U, \phi, \varepsilon)$, then $\{f_t^{-1}\} \in \mathscr{G}(U, \phi, \varepsilon)$.

c) Let $\{f_t\}$ and $\{g_t\}$ be elements of $\mathscr{G}(U, \phi, \varepsilon)$ such that there exists the composition $f_t \cdot g_t$ for $|t| < \varepsilon$. If $k \in C^\infty(U)$ and $|t_0| < \varepsilon$, then

$$\begin{aligned}&\frac{d}{dt}\bigg|_{t=t_0} k(f_t g_t)(g_{t_0}^{-1} f_{t_0}^{-1})(x) \\ &= \frac{d}{dt}\bigg|_{r=t_0} k f_r g_{t_0} g_{t_0}^{-1} f_{t_0}^{-1}(x) + \frac{d}{ds}\bigg|_{s=t_0} k f_{t_0} g_s g_{t_0}^{-1} f_{t_0}^{-1}(x) \\ &= v_{t_0}(x)(k) + (df_{t_0} u_{t_0})(x)(k) = (v_{t_0} + df_{t_0} u_{t_0})(x)(k) .\end{aligned}$$

From the first part of this lemma and Lemma 4.11,

$$v_{t_0} + df_{t_0} u_{t_0} \in G(U, \phi) ,$$

so that $\{f_t \cdot g_t\}$ is an element of $\mathscr{G}(U, \phi, \varepsilon)$.

5. The linear differential operators $B\phi$ and $L\phi$

5.1. Definition. Let M be a manifold, and μ, ω be two vector bundles over M. *A linear differential operator* from μ to ω is a C-linear map

$$P: C^\infty(\mu) \to C^\infty(\omega)$$

such that

$$\text{supp}\,(P(s)) \subset \text{supp}\,(s) \tag{14}$$

for any $s \in C^\infty(\mu)$.

The operator P is of *order* k if k is the smallest integer ($k \leq \infty$) such that for each $m \in M$ and each $f \in C^\infty(\mu)$ (see § 1.1),

$$j^k_m(f) = 0 \qquad \text{implies} \qquad P(f)(m) = 0\ .$$

5.2. We shall now recall Narasimhan's rewording [7] of a theorem due to Peetre: Let M be a manifold, μ and ω be two vector bundles over M, and P be a linear differential operator from μ to ω. Then every $m \in M$ has a neighbourhood U diffeomorphic to an open set Ω in R^n such that $\mu|U$ and $\omega|U$ are trivial and the induced operator from $\mu|U$ to $\omega|U$ over Ω has the form

$$\sum_{|\alpha| \leq k < \infty} a_\alpha(m) D^\alpha\ , \tag{15}$$

where if s, r are the ranks of μ and ω respectively, then the $a_\alpha(m)$'s are $s \times r$ matrices.

Remark. The theorem of Peetre says that for every $m \in M$ there is an open neighbourhood U of m where the order k of the operator is finite. Therefore the order will be finite in every relatively compact open set of M.

5.3. Let $(M; E, \pi, B)$ be a bundle of geometric objects, and ϕ a field of geometric objects defined on an open set U of M. If (TE, p', E) is the tangent bundle of E, then its restriction to $\pi^{-1}(U)$ is a new vector bundle, and the pull back of this vector bundle by ϕ [5, pp. 38, 39]

$$\begin{array}{ccc} \phi^*(T(\pi^{-1}(U))) & \xrightarrow{p'^*(\phi)} & T(\pi^{-1}(U)) \\ {\scriptstyle \phi^*(p')}\downarrow & & \downarrow{\scriptstyle p'} \\ U & \xrightarrow{\phi} & \pi^{-1}(U) \end{array}$$

is a vector bundle μ

$$\mu = \left(\bigcup_{x \in U} T_{\phi x}(\pi^{-1}(U)),\ \pi p,\ U\right)\ ,$$

where $\phi^*(p') = \pi p'$ with the corresponding restrictions.

If ξ is again the tangent bundle of M:

$$\xi = (TM, p, M) ,$$

we may define the operator

$$\boldsymbol{B}\phi\colon C^\infty(\xi \mid U) \to C^\infty(\mu)$$

by

$$\boldsymbol{B}\phi(v)(x) = \boldsymbol{B}(v)\phi_x$$

satisfying:

i) $\boldsymbol{B}\phi(v)$ is a section of μ,

ii) $\boldsymbol{B}\phi$ is a linear differential operator:

a) it is linear by Lemma 2.3,

b) it satisfies condition (14) of Definition 5.1. In fact, since $\operatorname{supp}(v) = \operatorname{Cl}\{x \in \operatorname{dom}(v); v(x) \neq 0\}$, the complement of $\operatorname{supp}(v)$ is open, and if z belongs to it there is a neighbourhood U_z of z such that $U_z \cap \operatorname{supp}(v) = \emptyset$. Then $v(y) = 0$ for every $y \in U_z$, and

$$\boldsymbol{B}\phi(v)(z) = \boldsymbol{B}(v)\phi_z = 0$$

by § 2.4 (z cannot be a frontier point).

Therefore the operator $\boldsymbol{B}\phi$ defined above is a linear differential operator, and by § 5.2 for each $x \in U \subset M$, there is an integer $0 \leq k < \infty$ such that

$$\boldsymbol{B}\phi(v)(x) = \sum_{|\alpha| < k < \infty} b_\alpha(x) D^\alpha(v)(x) \tag{16}$$

with the identifications $\xi_x \sim \boldsymbol{R}^n$, $\mu_x \sim \boldsymbol{R}^m$ ($n = \dim M$, $m = \dim E$).

We saw in § 1.5 **C** that the set

$$X = \bigcup_{x \in M} T(\pi^{-1}(x))$$

is a closed submanifold of TE, so that the vector bundle (X, p', E) is a vector subbundle of (TE, p', E). If ϕ is a field of geometric objects on an open set U of M, the pull back of (X, p', E) by ϕ:

$$\begin{array}{ccc} \phi^*(X) & \xrightarrow{p'^*(\phi)} & X \\ {\scriptstyle \phi^*(p')}\big\downarrow & & \big\downarrow{\scriptstyle p'} \\ U & \xrightarrow{\phi} & E \end{array}$$

is a new vector bundle ω, where $\phi^*(p') = \pi p' = \pi'$ with the corresponding restrictions,

$$\omega = \left(\bigcup_{x \in U} T_{\phi_x}(\pi^{-1}(x)),\, \pi',\, U \right) .$$

Therefore it is possible to define a new operator

$$L\phi\colon C^\infty(\xi \mid U) \to C^\infty(\omega)$$

by

$$L\phi(v)(x) = \underset{v}{L}\, \phi_x .$$

By Proposition 4.5.1 we know that

$$\underset{v}{L}\, \phi_x = d\phi(v) - \boldsymbol{B}(v)\phi_x$$

so that there exist $(n \times m)$-matrices a_α such that

$$\underset{v}{L}\, \phi_x = \sum_{|\alpha| \leq k < \infty} a_\alpha(x) D^\alpha(v)(x) . \tag{17}$$

From (17) it is clear that $L\phi$ is a linear differential operator which has the same order as $\boldsymbol{B}\phi$ at every $x \in U$.

5.4. Suppose that ϕ and ϕ' are two fields of geometric objects defined on an open set U of M with $\phi_{x_0} = \phi'_{x_0}$ for some $x_0 \in U$, and assume that for some open set W, $x_0 \in W \subset U$, there exist k and k' such that, for $x \in W$,

$$\boldsymbol{B}(v)\phi_x = \sum_{|\alpha| \leq k < \infty} b_\alpha(x) D^\alpha(v)(x) ,$$

$$\boldsymbol{B}(v)\phi'_x = \sum_{|\alpha| \leq k' < \infty} b'_\alpha(x) D^\alpha(v)(x) .$$

As $\phi_{x_0} = \phi'_{x_0}$, we have $\boldsymbol{B}(v)\phi_{x_0} = \boldsymbol{B}(v)\phi'_{x_0}$ and therefore

$$k = k' , \qquad b_\alpha(x_0) = b'_\alpha(x_0)$$

for each α, $|\alpha| \leq k = k'$.

5.5. Lemma. *Let $L\phi\colon C^\infty(\xi \mid U) \to C^\infty(\omega)$ be the linear differential operator defined in* § 5.3. *If the order of $L\phi$ is $k < \infty$ on an open set $W \subset U$, and $f\colon U \to f(U) \subset M$ is a diffeomorphism, then the order of $\boldsymbol{L}\boldsymbol{B}(f)\phi$ is also k.*

Proof. If the order of $L\phi$ on W is k, then k is the samallest integer such that for any $v \in C^\infty(\xi \mid U)$ and $x \in W$,

$$j^k_x(v) = 0 \qquad \text{implies} \qquad L\phi(v)(x) = 0 .$$

But being f a diffeomorphism, if $u \in C^\infty(\xi \mid f(U))$, then $j^k_{f(x)}(u) = 0$ implies $j^k_x(df^{-1}u) = 0$, and, by hypothesis.

$$0 = \underset{df^{-1}u}{L} \phi(x) = \underset{df^{-1}u}{L} \boldsymbol{B}(f^{-1})\boldsymbol{B}(f)\phi(x) = d\boldsymbol{B}(f^{-1})\, \underset{u}{L}\, (\boldsymbol{B}(f)\phi)(f(x)) .$$

Therefore by linearity,

$$j^k_{f(x)}(u) = 0 \qquad \text{implies} \qquad \underset{u}{L}\,(B(f)\phi)(f(x)) = 0 ,$$

where k is the smallest integer which satisfies this condition. If there is an integer $k' < k$ which satisfies the condition, it will be the order of $L\phi$.

5.6. Let, as before, $\xi = (TM, p, M)$, and let U be an open set of M such that U is homeomorphic to R^n and $\xi \mid U$ is trivial. It is possible to introduce a topology on $C^\infty(\xi \mid U)$ defining a fundamental system of neighbourhoods for every $v \in C^\infty(\xi \mid U)$ as follows [6]:

$$\mathscr{B}(m, v, k, \varepsilon) = \{u \in C^\infty(\xi \mid U) ; \| u - v \|^K_m < \varepsilon\} ,$$

where m is an arbitrary positive integer, K runs over all compact subsets of U and ε over all positive real numbers. We recall that the norms are given by

$$\| u \|^S_m = \sum_{|\alpha| \le m} \frac{1}{a!} \operatorname{supp}_{x \in S} |D^\alpha U(x)| \; (= \infty \text{ if } S \neq \text{compact}) ,$$

where $u \in C^\infty(\xi \mid U)$ and $S \subset U$. This topology satisfies:

a) if has a countable basis,

b) $\{v_i\} \to v$ if and only if $D^\alpha v_i \to D^\alpha v$ uniformly on compact sets for all α with $|\alpha| \le k$ (all α if $k = \infty$),

c) it is metrisable,

d) it is complete.

5.7. Now let W be an open subset of M, homeomorphic to an open subset of R^n such that $\xi \mid W$ and $\omega \mid W$, as defined in § 5.3, are trivial and

$$L\phi(v)(x) = \sum_{|\alpha| \le k < \infty} a_\alpha(x) D^\alpha(v)(x)$$

for $x \in W$ and $v \in C^\infty(\xi \mid W)$. By Peetre's theorem such W exists.

5.7.1. Lemma. *Let $\{v_i\}_{i \in I}$ be a sequence of vector fields of $C^\infty(\xi \mid W)$, such that $\{v_i\}_{i \in I} \to v$ uniformly on each compact subset $S \subset W$. If $\underset{v_i}{L}\,\phi_x = 0$ for some field ϕ of geometric objects defined on $W, x \in W$ and every $i \in I$, then $\underset{v}{L}\,\phi_x = 0$.*

Proof. Clearly from property b) of the topology of $C^\infty(\xi \mid W)$.

5.7.2. Lemma. *Let W and ϕ be the same elements defined above. Then*

$$G(W, \phi) = \{v, v \in C^\infty(\xi \mid W), \underset{v}{L}\,\phi_x = 0 \textit{ for each } x \in W\}$$

is a closed vector subspace of $C^\infty(\xi \mid W)$.

Proof. $G(W, \phi)$ is a vector subspace of $C^\infty(\xi \mid W)$ by Lemma 4.12. If $\{v_i\}$ is a sequence such that $v_i \in G(W, \phi)$ for every $i \in I$, and $\{v_i\} \to v$, then $v \in G(W, \phi)$ by Lemma 5.7.1.

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